Solution 8

1. Let f be continuously differentiable on $[a, b]$. Show that it has a differentiable inverse if and only if its derivative is either positive or negative everywhere. This is 2060 stuff.

   Solution. $\Rightarrow$. Let g be the inverse of f. When g is differentiable, we can use the chain rule in the relation $g(f(x)) = x$ to get $g'(f(x))f'(x) = 1$, which implies that $f'(x)$ never vanishes. Since f' is continuous, if $f'(x_0) > 0$ at some x_0, we claim f' is positive everywhere. Suppose $f'(x_1) < 0$ at some x_1, by continuity $f'(x_2) = 0$ at some x_2 between x_0 and x_1, contradiction holds. Hence f' is positive everywhere. Similarly, it is negative everywhere when it is negative at some point.

   $\Leftarrow$. Let us assume f' is always positive (the other case can be treated similarly.) Let $x < y$ in $[a, b]$. By the mean value theorem, there is some $z \in (x, y)$ such that $f(y) - f(x) = f'(z)(y - x) > 0$, so f is strictly increasing. According to an old result in 2050, a continuous, strictly increasing function maps $[a, b]$ to the interval $[f(a), f(b)]$ and its inverse g is continuous. Then we can use the Carathedory Criterion in 2060 to show that g is differentiable and, in fact, satisfies $g'(f(x)) = 1/f'(x)$.

2. Consider the function
$$f(x) = \frac{1}{2}x + x^2 \sin\frac{1}{x}, \quad x \neq 0,$$
   and set $f(0) = 0$. Show that f is differentiable at 0 with $f'(0) = 1/2$ but it has no local inverse at 0. Does it contradict the inverse function theorem?

   Solution. $|f(x) - f(0) - (1/2)x| = |x^2 \sin(1/x)| = O(x^2)$, hence f is differentiable at 0 with $f'(0) = 1/2$. Let $x_k = 1/2k\pi, y_k = 1/(2k\pi + 1)$, then $f'(x_k) = -1/2, f'(y_k) = 3/2$. Then it is clear that f is not injective in $I_k = (y_k, x_k)$. Since any neighborhood of 0 must include contain some I_k, this shows that f it has no local inverse at 0. It does not contradict the inverse function theorem because f' is not continuous at 0.

   Note. This problem shows that the C^1-condition is needed in the Inverse Function Theorem.

3. Study the map on $\mathbb{R}^2$ given $(x, y) \mapsto (x^2 - y^2, 2xy)$. Show that it is local invertible everywhere except at the origin. Does its inverse exist globally?

   Solution. In terms of complex notation, the map is simply $F(z) = z^2$. Its inverse is $G(z) = z^{1/2}$. The preimage of F contains two points $\sqrt{r}e^{i\theta/2}$ and $-\sqrt{r}e^{i\theta/2}$ where $z = re^{i\theta},\ \theta \in [0, 2\pi)$.

4. Consider the mapping from $\mathbb{R}^2$ to itself given by $f(x, y) = x - x^2,\ g(x, y) = y + xy$. Show that it has a local inverse at $(0, 0)$. And then write down the inverse map so that its domain can be described explicitly.

   Solution. Let $u = x - x^2, v = y + xy$. The Jacobian determinant is 1 at $(0, 0)$ so there is an inverse in some open set containing $(0, 0)$. Now we can describe it explicitly as follows. From the first equation we have
$$x = \frac{1 \pm \sqrt{1 - 4u}}{2}.$$
   From $u(0, 0) = 0$ we must have
$$x = \frac{1 - \sqrt{1 - 4u}}{2}\ .$$

Then

$$y = \frac{v}{1+x} = \frac{2v}{3-\sqrt{1-4u}}.$$

We see that the largest domain in which the inverse exists is $\{(u,v) : u \in (-2, 1/4), v \in \mathbb{R}\}$.

5. Let F be a continuously differentiable map from the open $U \subset \mathbb{R}^n$ to $\mathbb{R}^n$ whose Jacobian determinant is non-vanishing everywhere. Prove that it maps every open set in U to an open set, that is, F is an open map. Does its inverse $F^{-1} : F(U) \to U$ always exist?

   Solution. Let E be an open set in U. We need to show that $F(E)$ is open. Let $y_0 \in F(E)$ and $x_0 \in E$ satisfy $F(x_0) = y_0$. By the Inverse Function Theorem (applied to $F : E \to \mathbb{R}^n$), there are open sets V (in E) and W containing x_0 and y_0 respectively such that $F(V) = W$. In particular, $W \subset F(E)$. Since W is open and contains y_0, there is some $B_r(y_0) \subset W \subset F(E)$, so $F(E)$ is open.

   The inverse may not exist. Consider the map $(r,\theta) \to (r\cos\theta, r\sin\theta)$ in $(r,\theta) \in (0,\infty)\times\mathbb{R}$, whose Jacobian determinant is always nonzero. However, it has no inverse.

6. Consider the function

$$h(x,y) = (x-y^2)(x-3y^2), \quad (x,y) \in \mathbb{R}^2.$$

   Show that the set $\{(x,y) : \ h(x,y) = 0\}$ cannot be expressed as a local graph of a C^1-function over the x or y-axis near the origin. Explain why the Implicit Function Theorem is not applicable.

   Solution. The Jacobian matrix of h is singular at $(0,0)$, hence the Implicit Function Theorem cannot apply. Indeed, $h(x,y) = 0$ means either $x - y^2 = 0$ or $x - 3y^2 = 0$. The solution set of $\{(x,y) : \ h(x,y) = 0\}$ consisting of two different parabolas passing the origin.

7. Consider a real polynomial $p(x,\mathbf{a}) = a_0 + a_1 x + \cdots + a_n x^n$ as a function of x and its coefficients. A point x_0 is a simple root of p if $p(x_0,\mathbf{a}) = 0$ and $p'(x_0,\mathbf{a}) \neq 0$ where $\mathbf{a} = (a_0, a_1, \cdots, a_n)$. Let x_0 be a simple of $p(\cdot,\mathbf{a}_0)$. Show that there is a smooth function φ defined in an open set in $\mathbb{R}^{n+1}$ containing $\mathbf{a}_0$ such that $x = \varphi(\mathbf{a})$ is a simple root for $p(\cdot,\mathbf{a}) = 0$. What happens when root is not simple?

   Solution. This is a straightforward application of the Implicit Function Theorem. Consider the quadratic equation $ax^2 + bx + c = 0$ whose solutions are given by $(-b \pm \sqrt{b^2 - 4ac})/2a$. The root is not simple when $b^2 - 4ac = 0$. When the coefficients vary, the roots could be simple, not simple and imaginary. Therefore its inverse does not exist locally.